Structure of some indecomposable groups over finite field

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Abstract

The objective of this paper is to give a complete set of primitive central idempotents and the Wedderburn decomposition of certain classes of semisimple group algebras over finite field.

Keywords: *semisimple group algebras, metabelian groups , indecomposable groups, primitive central idempotents, Wedderburn decomposition.*

MSC2000: 16S34; 20C05; 16K20

1. Introduction

The problem of finding a complete set of primitive central idempotents of semisimple group algebra has attracted the attention of various authors **[1-8,16-18]**. The knowledge of complete set of primitive central idempotents is useful in finding the Wedderburn decomposition, unit group, automorphism group and in error correcting codes **[12-15,19,20]**. Bakshi et. al. **[3,4,5]** have solved this problem for semisimple group algebra $F_q[G]$, where F_q is a finite field with q elements and G is a finite metabelian group, using Strong Shoda pairs.

Let G be a finite indecomposable group with its central quotient Klein's four group. Milies et. al.**[9,10,11]** have classified all such groups in five classes $D_i, i = 1, \dots, 5$. In this paper we will find the complete algebraic structure of semisimple group algebra $F_q[G]$, G of type D_4, given by the presentation

$$a, b, x, y \vee a^2 = x, b^2 = y, x^{2^{m_1}} = 1, y^{2^{m_2}} = 1, a^{-1}b^{-1}ab = a^{2^{m_1}} >, \; m_1, m_2 \geq 1,$$

using the method developed in **[5]**.

Notation

Let F_q be a finite field with q elements and let $\acute{F}_q$ be its algebraic closure. Let $H \trianglelefteq K \trianglelefteq G$ with K/H cyclic of order n. Set

ζ	a primitive nth root of unity in $\acute{F}_q$
$o_n(q)$	order of q modulo n
G'	the derived subgroup of G
$[G:H]$	index of H in G
$N_G(H)$	the normalizer of H in G
$Irr(K/H)$	the set of irreducible characters of K/H over $\acute{F}_q$
$\mathcal{C}(K/H)$	set of q-cyclotomic cosets of $Irr(K/H)$ containing the generators of

	$Irr(K/H)$
$E_G(K/H)$	stabilizer of $C \in \mathcal{C}(K/H)$ under the action of $N_G(H)$ on $\mathcal{C}(K/H)$ by conjugation
$\mathcal{R}(K/H)$	set of distinct orbits of $\mathcal{C}(K/H)$ under the above action of $N_G(H)$ on $\mathcal{C}(K/H)$
$\varepsilon_C(K,H)$	$\acute{g}$ $\chi(\ \ )$ $C \in \mathcal{C}(K/H)$ $tr_{F_q(\zeta)/F_q}$ $\lvert K\rvert^{-1} \sum_{g\in K}$
$e_C(G,K,H)$	sum of distinct G -conjugates of $\varepsilon_C(K,H)$

For $N \trianglelefteq G$, let

A_N/N	a maximal abelian subgroup of G/N containing $(G/N)'$
T	set of all subgroups D/N of G/N with $D/N \trianglelefteq A_N/N$, A_N/D cyclic

Consider the action of conjugation in G/N. Set

$T_{G/N}$	set of representatives of the distinct equivalence classes of T
$S_{G/N}$	$\{(D/N\,, A_N/N) \mid D/N \in T_{G/N,}\, D/N$core free in G/N
S	$\{(N, D/N\,, A_N/N) \mid N \trianglelefteq G, S_{G/N} \neq \emptyset, (D/N\,, A_N/N) \in S_{G/N}$

Theorem 1 [5] Let F_q be a finite field with q elements and G a finite metabelian group. Suppose that $gcd(q, |G|) = 1$. Then a complete set of primitive central idempotents of F_q[G] is given by the set

$$\{e_C(G, A_N, D) \mid (N, D/N\,, A_N/N) \in S, C \in \mathcal{R}_{(N|\quad |D)}^{A}.$$

Moreover, the corresponding simple component $F_q[G]e_c(G, A_N, D)$ is isomorphic to $M_{[G:A_N]}\left(F_{q^{o(A_N,D)}}\right)$, the algebra of $[G: A_N] \times [G: A_N]$ matrices over the field $F_{q^{o(A_N,D)}}$, where $o(A_N, D) = \frac{o_{[A_N:D]}(q)}{[E_G(A_N/D):A_N]}$.

We are now ready to give the complete algebraic structure of $F_q[G]$, G of type D_4.

1. Groups G of type D_4

Let $G \coloneqq D_4 = a, b, x, y \vee a^2 = x, b^2 = y, x^{2^{m_1}} = 1, y^{2^{m_2}} = 1, a^{-1}b^{-1}ab = a^{2^{m_1}} >$
Clearly G is a metacyclic group generated by a, b with the relations $a^{2^{m_1+1}} = 1 = b^{2^{m_2+1}}$, $b^{-1}ab = a^{2^{m_1+1}}$, a^2, b^2 central in G.

Let $\eta = \{(2^\alpha, 0, 2^\gamma) | 0 \le \alpha \le m_1, 0 \le \gamma \le m_2 + 1\} \cup \{(2^\alpha, i, 2^\gamma) | 1 \le i \le 2^\alpha - 1,$

$1 \le \alpha \le m_1, \gcd(i, 2^\alpha) = 2^\beta, 0 \le \beta \le \alpha - 1, 0 \le \gamma \le m_2 + 1 - \alpha + \beta \cup$

$\{(2^{m_1+1}, 0, 2^\gamma) | 1 \le \gamma \le m_2 + 1\} \cup \{(2^{m_1+1}, i, 2^\gamma) \vee \gcd(i, 2^{m_1+1}) = 2^\beta,$

$0 \le \beta \le m_1, \; 1 \le \gamma \le m_2 - m_1 + \beta$

$= A \cup B \cup C \cup D$ (say).

If follows from **[5, Lemma 1]**, that $H_{v,i,c} = \langle a^v, a^i b^c \rangle$, $(v, i, c) \in \eta$, are all the distinct normal subgroups of G.

Theorem 2 Let $m_1, m_2 \ge 1$. Then for $m_1 > m_2$, the complete algebraic structure of semisimple group algebra $F_q[G]$, G of type D_4, is as follows:

Primitive central idempotents

$e_C(G, G, G), C \in \mathfrak{R}(G/G)$;

$e_C(G, G, \langle a^{2^\alpha}, b \rangle), C \in \mathfrak{R}(G/\langle a^{2^\alpha}, b \rangle)$;

$e_C(G, G, \langle a, b^{2^\gamma} \rangle), C \in \mathfrak{R}(G/\langle a, b^{2^\gamma} \rangle)$;

$e_C(G, G, \langle a^{2^\alpha}, a^i b^{2^\gamma} \rangle), C \in \mathfrak{R}(G/\langle a^{2^\alpha}, a^i b^{2^\gamma} \rangle)$, $\gcd(i, 2^\alpha) = 1$,

$1 \le \gamma \le m_2 + 1 - \alpha$;

$e_C(G, G, \langle a^{2^\alpha}, a^i b \rangle), C \in \mathfrak{R}(G/\langle a^{2^\alpha}, a^i b \rangle)$, $\gcd(i, 2^\alpha) = 2^\beta$,

where $\begin{cases} 0 \le \beta \le \alpha - 1, 1 \le \alpha \le m_2 + 1, \\ \alpha - m_2 - 1 \le \beta \le \alpha - 1, m_2 + 2 \le \alpha \le m_1 - 1. \end{cases}$

$e_C(G, \langle a, b^2 \rangle, \langle b^2 \rangle), C \in \mathfrak{R}(\langle a, b^2 \rangle / \langle b^2 \rangle)$;

$e_C(G, \langle a, b^2 \rangle, \langle a^i b^2 \rangle), C \in \mathfrak{R}(\langle a, b^2 \rangle / \langle a^i b^2 \rangle)$, $\gcd(i, 2^{m_1+1}) = 2^\beta$,

$m_1 - m_2 + 1 \le \beta \le m_1$.

Wedderburn decomposition

$$F_q \oplus_{\alpha=1}^{m_1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{\alpha-1}}{f_\alpha}\right)} \oplus_{\gamma=1}^{m_2+1} \left(F_{q^{f_\gamma}}\right)^{\left(\frac{2^{\gamma-1}}{f_\gamma}\right)} \oplus_{\alpha=1}^{m_2+1} \oplus_{\beta=0}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)}$$
$$\oplus_{\alpha=m_2+2}^{m_1} \oplus_{\beta=\alpha-m_2-1}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)} \oplus_{\alpha=1}^{m_2+1} \oplus_{\gamma=1}^{m_2+1-\alpha} \left(F_{q^{f_{\alpha+\gamma}}}\right)^{\left(\frac{2^{2\alpha+\gamma-2}}{f_{\alpha+\gamma}}\right)}$$
$$\oplus M_2\left(F_{q^{f_{m_1+1}}}\right)^{\left(\frac{2^{m_1-1}}{f_{m_1+1}}\right)} \oplus_{\beta=m_1-m_2+1}^{m_1} M_2\left(F_{q^{f_{m_1+1}}}\right)^{\left(\frac{2^{2m_1-\beta-1}}{f_{m_1+1}}\right)}, 2 \nmid f_{m_1+1},$$
$$F_q \oplus_{\alpha=1}^{m_1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{\alpha-1}}{f_\alpha}\right)} \oplus_{\gamma=1}^{m_2+1} \left(F_{q^{f_\gamma}}\right)^{\left(\frac{2^{\gamma-1}}{f_\gamma}\right)} \oplus_{\alpha=1}^{m_2+1} \oplus_{\beta=0}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)}$$
$$\oplus_{\alpha=m_2+2}^{m_1} \oplus_{\beta=\alpha-m_2-1}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)} \oplus_{\alpha=1}^{m_2+1} \oplus_{\gamma=1}^{m_2+1-\alpha} \left(F_{q^{f_{\alpha+\gamma}}}\right)^{\left(\frac{2^{2\alpha+\gamma-2}}{f_{\alpha+\gamma}}\right)}$$
$$\oplus M_2\left(F_{q^{\frac{f_{m_1+1}}{2}}}\right)^{\left(\frac{2^{m_1}}{f_{m_1+1}}\right)} \oplus_{\beta=m_1-m_2+1}^{m_1} M_2\left(F_{q^{\frac{f_{m_1+1}}{2}}}\right)^{\left(\frac{2^{2m_1-\beta}}{f_{m_1+1}}\right)}, 2 \vee f_{m_1+1},$$

$$F_q[G] \cong$$

where $f_i = o_{2^i}(q)$, the order of q modulo 2^i, $i \geq 1$.

Proof: To find the complete algebraic structure of $F_q[G]$, G of type D_4, we will find $S_{G/N}$ for each normal subgroup N of G.

Observe that for $N = H_{v,i,c}$, $(v,i,c) \in A \cup B$, $G' = a^{2^{m_1}} > \subseteq N$, thus G/N is abelian and hence

$$S_{G/N} = \begin{cases} (1 >, G/N), if\ G/N\ iscyclic, \\ \emptyset, otherwise. \end{cases}$$

It can be seen easily that for $N = H_{v,i,c}$, $(v,i,c) \in A \cup B$ the following have cyclic quotients with G:

(i) $a^{2^\alpha}, b >, 0 \leq \alpha \leq m_1$;

(ii) $a, b^{2^\gamma} >, 1 \leq \gamma \leq m_2 + 1$;

(iii) $a^{2^\alpha}, a^i b >, 1 \leq \alpha \leq m_1$, $\gcd(i, 2^\alpha) = 2^\beta$, where

$$\begin{cases} 0 \leq \beta \leq \alpha - 1, 1 \leq \alpha \leq m_2 + 1, \\ \alpha - m_2 - 1 \leq \beta \leq \alpha - 1, m_2 + 2 \leq \alpha \leq m_1; \end{cases}$$

(iv) $a^{2^\alpha}, a^i b^{2^\gamma} >, 1 \leq \alpha \leq m_2 + 1$, $\gcd(i, 2^\alpha) = 1$, $1 \leq \gamma \leq m_2 + 1 - \alpha$.

For all these N, $S_{G/N} = (1 >, G/N)$.

Table

N	(D, A_N)	$o(A_N, D)$	$\lvert \mathcal{R}(A_N, D) \vee$
$a, b >$	(N, G)	1	1
$a^{2^\alpha}, b >$ $0 \leq \alpha \leq m_1$	(N, G)	f_α	$\frac{2^{\alpha-1}}{f_\alpha}$

$a, b^{2^\gamma} >$ $1 \le \gamma \le m_2 + 1$	(N, G)	f_γ	$\frac{2^{\gamma-1}}{f_\gamma}$
$a^{2^\alpha}, a^i b >$ $\gcd(i, 2^\alpha) = 2^\beta, 1 \le \alpha \le m_1$ $\begin{cases} 0 \le \beta \le \alpha - 1, 1 \le \alpha \le m_2 + 1, \\ \alpha - m_2 - 1 \le \beta \le \alpha - 1, m_2 + 2 \le \alpha \le m_1 \end{cases}$	(N, G)	f_α	$\frac{2^{\alpha-1}}{f_\alpha}$
$a^{2^\alpha}, a^i b^{2^\gamma} >, 1 \le \alpha \le m_2 + 1,$ $\gcd(i, 2^\alpha) = 1,\ 1 \le \gamma \le m_2 + 1 - \alpha$	(N, G)	$f_{\alpha+\gamma}$	$\frac{2^{\alpha+\gamma-1}}{f_{\alpha+\gamma}}$

Now for $N = H_{v,i,c} \in C \cup D,\ S_{G/N} \ne \emptyset \Leftrightarrow N = b^2 >$ or $a^i b^2 >,\ \ \gcd(i, 2^{m_1+1}) = 2^\beta,$ $m_1 - m_2 + 1 \le \beta \le m_1$.

Moreover, for $N = b^2 >$, $S_{G/N} = \frac{a, b^2 > \ /N}{1 >, <}$ and for $N = a^i b^2 >$, $S_{G/N} = \frac{a, b^2 > \ /N}{1 >, <}$.

Primitive central idempotents, stated in Theorem 2 are thus obtained with the help of Theorem 1.

Table

N	(D, A_N)	$o(A_N, D)$	$\lvert\mathcal{R}(A_N, D) \vee$
$b^2 >$	$N, < a, b^2 >$	$\begin{cases} f_{m_1+1}, 2 \nmid f_{m_1+1}, \\ \frac{f_{m_1+1}}{2}, 2 \vee f_{m_1+1}. \end{cases}$	$\begin{cases} \frac{2^{m_1-1}}{f_{m_1+1}}, 2 \nmid f_{m_1+1}, \\ \frac{2^{m_1}}{f_{m_1+1,}}, 2 \vee f_{m_1+1}. \end{cases}$
$a^i b^2 >$ $\gcd(i, 2^{m_1+1}) = 2^\beta$ $m_1 - m_2 + 1 \le \beta \le m_1$	$N, < a, b^2 >$	$\begin{cases} f_{m_1+1}, 2 \nmid f_{m_1+1}, \\ \frac{f_{m_1+1}}{2}, 2 \vee f_{m_1+1}. \end{cases}$	$\begin{cases} \frac{2^{m_1-1}}{f_{m_1+1}}, 2 \nmid f_{m_1+1}, \\ \frac{2^{m_1}}{f_{m_1+1,}}, 2 \vee f_{m_1+1}. \end{cases}$

The Wedderburn decomposition of $F_q[G]$, G of type D_4 is thus obtained with the help of Tables 1, 2 and Theorem 1.

Theorem 3 (i) For $m_1 < m_2$, the complete algebraic structure of semisimple group algebra $F_q[G]$, G of type D_4, is as follows:

Primitive central idempotents

$e_C(G, G, G), C \in \mathcal{R}(G/G)$;

$G, G, a^{2^\alpha}, b >$ $\quad a^{2^\alpha}, b >$

$\mathcal{R}$ $G/$;

e_C

$G, G, a, b^{2^\gamma} >$ $\quad a, b^{2^\gamma} >$

$\mathcal{R}$ $G/$;

e_C

$G, G, a^{2^\alpha}, a^i b^{2^\gamma} >$ $\quad a^{2^\alpha}, a^i b^{2^\gamma} >$

$\mathcal{R}$ $G/$, $\gcd(i, 2^\alpha) = 1$,

e_C

$1 \leq \gamma \leq m_2 + 1 - \alpha$;

$G, G, a^{2^\alpha}, a^i b >$ $\quad a^{2^\alpha}, a^i b >$

$\mathcal{R}$ $G/$, $\gcd(i, 2^\alpha) = 2^\beta$,

e_C

$0 \leq \beta \leq \alpha - 1$;

$G, < a, b^2 >, b^2 >$ $\quad b^2 >$

$\mathcal{R}^{a, b^2 >}$ $/$;

e_C

$G, < a, b^2 >, a^i b^2 >$ $\quad a^i b^2 >$

$\mathcal{R}^{a, b^2 >}$ $/$, $\gcd(i, 2^{m_1+1}) = 2^\beta$,

e_C

$1 \leq \beta \leq m_1$;

$G, < a, b^2 >, a^{\frac{i}{2}} b^{2^{\gamma-1}} >$ $\quad a^{\frac{i}{2}} b^{2^{\gamma-1}} >$

$\mathcal{R}^{a, b^2 >}$ $/$, $\gcd(i, 2^{m_1+1}) = 2$,

e_C

$2 \leq \gamma \leq m_2 - m_1 + 1$.

Wedderburn decomposition

$$F_q[G] \cong \begin{cases} F_q \oplus_{\alpha=1}^{m_1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{\alpha-1}}{f_\alpha}\right)} \oplus_{\gamma=1}^{m_2+1} \left(F_{q^{f_\gamma}}\right)^{\left(\frac{2^{\gamma-1}}{f_\gamma}\right)} \oplus_{\alpha=1}^{m_1} \oplus_{\beta=0}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)} \\ \oplus_{\alpha=1}^{m_1} \oplus_{\gamma=1}^{m_2+1-\alpha} \left(F_{q^{f_{\alpha+\gamma}}}\right)^{\left(\frac{2^{2\alpha+\gamma-2}}{f_{\alpha+\gamma}}\right)} \oplus M_2\left(F_{q^{f_{m_1+1}}}\right)^{\left(\frac{2^{m_1-1}}{f_{m_1+1}}\right)} \\ \oplus_{\beta=1}^{m_1} M_2\left(F_{q^{f_{m_1+1}}}\right)^{\left(\frac{2^{2m_1-\beta-1}}{f_{m_1+1}}\right)} \oplus_{\gamma=2}^{m_2-m_1+1} M_2\left(F_{q^{f_{m_1+\gamma-2}}}\right)^{\left(\frac{2^{2m_1+\gamma-3}}{f_{m_1+\gamma-2}}\right)}, 2 \nmid f_{m_1+1}, \\ F_q \oplus_{\alpha=1}^{m_1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{\alpha-1}}{f_\alpha}\right)} \oplus_{\gamma=1}^{m_2+1} \left(F_{q^{f_\gamma}}\right)^{\left(\frac{2^{\gamma-1}}{f_\gamma}\right)} \oplus_{\alpha=1}^{m_1} \oplus_{\beta=0}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)} \\ \oplus_{\alpha=1}^{m_1} \oplus_{\gamma=1}^{m_2+1-\alpha} \left(F_{q^{f_{\alpha+\gamma}}}\right)^{\left(\frac{2^{2\alpha+\gamma-2}}{f_{\alpha+\gamma}}\right)} \oplus M_2\left(F_{q^{\frac{f_{m_1+1}}{2}}}\right)^{\left(\frac{2^{m_1}}{f_{m_1+1}}\right)} \\ \oplus_{\beta=1}^{m_1} M_2\left(F_{q^{\frac{f_{m_1+1}}{2}}}\right)^{\left(\frac{2^{2m_1-\beta}}{f_{m_1+1}}\right)} \oplus_{\gamma=2}^{m_2-m_1+1} M_2\left(F_{q^{f_{m_1+\gamma-2}}}\right)^{\left(\frac{2^{2m_1+\gamma-3}}{f_{m_1+\gamma-2}}\right)}, 2 \vee f_{m_1+1}. \end{cases}$$

(ii) For $m_1 = m_2$, the complete algebraic structure of semisimple group algebra $F_q[G]$, G of type D_4, is as follows:

Primitive central idempotents

$e_C(G,G,G), C \in \mathscr{R}(G/G)$;

$G, G, a^{2^\alpha}, b >$ $a^{2^\alpha}, b >$
$\mathscr{R}$ $G/$;
e_C

$G, G, a, b^{2^\gamma} >$ $a, b^{2^\gamma} >$
$\mathscr{R}$ $G/$;
e_C

$G, G, a^{2^\alpha}, a^i b^{2^\gamma} >$ $a^{2^\alpha}, a^i b^{2^\gamma} >$
$\mathscr{R}$ $G/$, $\gcd(i, 2^\alpha) = 1$,
e_C

$1 \le \gamma \le m_1 + 1 - \alpha$;

$G, G, a^{2^\alpha}, a^i b >$ $a^{2^\alpha}, a^i b >$
$\mathscr{R}$ $G/$, $\gcd(i, 2^\alpha) = 2^\beta$,
e_C

$0 \le \beta \le \alpha - 1$;

$G, < a, b^2 >, b^2 >$ $b^2 >$
$\mathscr{R} a, b^2 >$ / ;
e_C

$G, < a, b^2 >, a^i b^2 >$ $a^i b^2 >$
$\mathscr{R} a, b^2 >$ / , $\gcd(i, 2^{m_1+1}) = 2^\beta$,
e_C

$1 \le \beta \le m_1$.

Wedderburn decomposition

$$F_q[G] \cong \begin{cases} F_q \oplus_{\alpha=1}^{m_1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{\alpha-1}}{f_\alpha}\right)} \oplus_{\gamma=1}^{m_1+1} \left(F_{q^{f_\gamma}}\right)^{\left(\frac{2^{\gamma-1}}{f_\gamma}\right)} \oplus_{\alpha=1}^{m_1} \oplus_{\beta=0}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)} \\ \quad \oplus_{\alpha=1}^{m_1} \oplus_{\gamma=1}^{m_1+1-\alpha} \left(F_{q^{f_{\alpha+\gamma}}}\right)^{\left(\frac{2^{2\alpha+\gamma-2}}{f_{\alpha+\gamma}}\right)} \oplus M_2\left(F_{q^{f_{m_1+1}}}\right)^{\left(\frac{2^{m_1-1}}{f_{m_1+1}}\right)} \\ \quad \oplus_{\beta=1}^{m_1} M_2\left(F_{q^{f_{m_1+1}}}\right)^{\left(\frac{2^{2m_1-\beta-1}}{f_{m_1+1}}\right)}, 2 \nmid f_{m_1+1}, \\ F_q \oplus_{\alpha=1}^{m_1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{\alpha-1}}{f_\alpha}\right)} \oplus_{\gamma=1}^{m_1+1} \left(F_{q^{f_\gamma}}\right)^{\left(\frac{2^{\gamma-1}}{f_\gamma}\right)} \oplus_{\alpha=1}^{m_1} \oplus_{\beta=0}^{\alpha-1} \left(F_{q^{f_\alpha}}\right)^{\left(\frac{2^{2\alpha-\beta-2}}{f_\alpha}\right)} \\ \quad \oplus_{\alpha=1}^{m_1} \oplus_{\gamma=1}^{m_1+1-\alpha} \left(F_{q^{f_{\alpha+\gamma}}}\right)^{\left(\frac{2^{2\alpha+\gamma-2}}{f_{\alpha+\gamma}}\right)} \oplus M_2\left(F_{q^{\frac{f_{m_1+1}}{2}}}\right)^{\left(\frac{2^{m_1}}{f_{m_1+1}}\right)} \\ \quad \oplus_{\beta=1}^{m_1} M_2\left(F_{q^{\frac{f_{m_1+1}}{2}}}\right)^{\left(\frac{2^{2m_1-\beta}}{f_{m_1+1}}\right)}, 2 \vee f_{m_1+1}. \end{cases}$$

Proof: (i) As in Theorem 2, it can be seen easily that for $N = H_{v,i,c}$, $(v,i,c) \in A \cup B$, $S_{G/N} \neq \emptyset \Leftrightarrow N$ is of the following type:

(i) $a^{2^\alpha}, b >,\ 0 \leq \alpha \leq m_1;$

(ii) $a, b^{2^\gamma} >,\ \ 1 \leq \gamma \leq m_2 + 1;$

(iii) $a^{2^\alpha}, a^i b >,\ \ 1 \leq \alpha \leq m_1,\ \gcd(i, 2^\alpha) = 2^\beta,\ 0 \leq \beta \leq \alpha - 1;$

(iv) $a^{2^\alpha}, a^i b^{2^\gamma} >,\ 1 \leq \alpha \leq m_1,\ \gcd(i, 2^\alpha) = 1,\ 0 \leq \gamma \leq m_2 + 1 - \alpha.$

For $N = H_{v,i,c}, (v,i,c) \in C \cup D, S_{G/N} \neq \emptyset \Leftrightarrow N$ is of the following types:

(i) $b^2 >;$

(ii) $a^i b^2 >,\ \ \gcd(i, 2^{m_1+1}) = 2^\beta, 1 \leq \beta \leq m_1;$

(iii) $a^i b^{2^\gamma} >,\ \gcd(i, 2^{m_1+1}) = 2,\ 2 \leq \gamma \leq m_2 - m_1 + 1.$

For $N = b^2 >$, $a, b^2 >\ /N$ $1 >, <$,

$S_{G/N} =$

for $N = a^i b^2 >$, $a, b^2 >\ /N$ $1 >, <$,

$S_{G/N} =$

for $N = a^i b^{2^\gamma} >$, $a, b^2 >\ /N$ $a^{\frac{i}{2}} b^{2^{\gamma-1}} >\ /N, <.$

$S_{G/N} =$

Table

N	(D, A_N)	$o(A_N, D)$	$\vert\mathcal{R}(A_N, D) \vee$
$a, b >$	(N, G)	1	1
$a^{2^\alpha}, b >$ $0 \leq \alpha \leq m_1$	(N, G)	f_α	$\frac{2^{\alpha-1}}{f_\alpha}$
$a, b^{2^\gamma} >$ $1 \leq \gamma \leq m_2 + 1$	(N, G)	f_γ	$\frac{2^{\gamma-1}}{f_\gamma}$
$a^{2^\alpha}, a^i b^{2^\gamma} >,$ $1 \leq \alpha \leq m_2 + 1,$ $\gcd(i, 2^\alpha) = 1,$ $1 \leq \gamma \leq m_2 + 1 - \alpha$	(N, G)	$f_{\alpha+\gamma}$	$\frac{2^{\alpha+\gamma-1}}{f_{\alpha+\gamma}}$
$a^{2^\alpha}, a^i b >$ $\gcd(i, 2^\alpha) = 2^\beta,$ $1 \leq \alpha \leq m_1$ $0 \leq \beta \leq \alpha - 1$	(N, G)	f_α	$\frac{2^{\alpha-1}}{f_\alpha}$
$b^2 >$	$b^2 >, < a, b^2 >$	$\begin{cases} f_{m_1+1}, 2 \nmid f_{m_1+1}, \\ \frac{f_{m_1+1}}{2}, 2 \vee f_{m_1+1}. \end{cases}$	$\begin{cases} \frac{2^{m_1-1}}{f_{m_1+1}}, 2 \nmid f_{m_1+1}, \\ \frac{2^{m_1}}{f_{m_1+1,}}, 2 \vee f_{m_1+1}. \end{cases}$
$a^i b^2 >$ $\gcd(i, 2^{m_1+1}) = 2^\beta$ $1 \leq \beta \leq m_1$	$N, < a, b^2 >$	$\begin{cases} f_{m_1+1}, 2 \nmid f_{m_1+1}, \\ \frac{f_{m_1+1}}{2}, 2 \vee f_{m_1+1}. \end{cases}$	$\begin{cases} \frac{2^{m_1-1}}{f_{m_1+1}}, 2 \nmid f_{m_1+1}, \\ \frac{2^{m_1}}{f_{m_1+1,}}, 2 \vee f_{m_1+1}. \end{cases}$
$a^i b^{2^\gamma} >$ $\gcd(i, 2^{m_1+1}) = 2,$ $2 \leq \gamma \leq m_2 - m_1 + 1$	$a^{\frac{i}{2}} b^{2^{\gamma-1}} >, < a, b^2 >$	$f_{m_1+\gamma-1}$	$\frac{2^{m_1+\gamma-2}}{f_{m_1+\gamma-1}}$

The Primitive central idempotents and Wedderburn decomposition of $F_q[G]$, G of type D_4 is thus obtained with the help of Table 3 and Theorem 1.

The proof of (ii) is similar to the previous one.

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