A Survey of Lightlike Submersions

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Abstract

This is a survey article with main object to present the main results related to Riemannian submersions, lightlike submersions, screen lightlike submersions, screen conformal lightlike submersions and harmonic maps from lightlike manifolds.

Keywords: *Riemannian submersion, lightlike submersion, screen lightlike submersion, screen conformal lightlike submersions,harmonic maps.*

MSC2010: 53C20, 53C50.

1. Introduction

Submersions: The dictionary meaning of a submersion is to go below or make something go below the surface of the sea or a river or to cover or hide something completely. But in geometry by submersions we mean certain kind of mapping which is used to compare the geometrical structures such as curvature of two manifolds or to transfer some structure from one manifold to another manifold. In spite of the fact that the theory of submersion and immersions are totally different, submersions in some sense can be thought as the counterpart of immersion. The study of Riemannian submersion $\varphi: M \to B$ was initiated by O'Neill [15]. The projection of a Riemannian product manifold on one of its factor is the trivial example of Riemannian submersion. Later semi-Riemannian submersions were introduced by O'Neill in [16].Horizontally conformal submersions came into existence and after that Sahin in[17] introduced Screen conformal lightlike submersion which can be considered as a lightlike version of horizontally conformal submersions. A projection from warped product lightlike manifold onto its second factor is a simple example of screen conformal lightlike submersion. Moreover a screen lightlike submersion is a screen conformal lightlike subersion with dilation identically one. In case of Riemannian submersion fibers are always Riemannian but in case of semi-Riemannian submersions the fibers need not be Riemannian(hence semi-Riemannian) manifolds. Thus lightlike submersions from a lightlike manifold onto a semi-Riemannian manifold were introduced by Sahin[18], as a lightlike version of Riemannian submersion. Semi-Riemannian submersions are of interest in Mathematical-physics, owing to applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories. In Kaluza-Klein theory, the general solution of a recent model can be expressed in terms of harmonic maps satisfying Einstein equations,(for details see [5,6,10,11,21]). Altafini [1] presented some applications of submersion in the theory of robotics. Sahin [20] also gave some applications of Riemannian submersions on redundant robotic chains obtained by

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Altafini. Submersions also have their applications in statistical analysis on manifolds [4], statistical machine learning process [22] and medical imaging [14].

Definition 1.1. **([7])**. Let M and N be smooth manifolds with dimensions m and n respectively. Let f : M → N be a smooth map and f_* : $T_pM \to T_{f(p)}N$ be a tangential map at p ∈ M, then f is said to be an immersion if f_* is injective and f is said to be submersion if f_* is surjective for all p ∈ M

2. Riemannian Submersions

Definition 2.1. ([11]). Let (M, g) and (B, h) be C^∞-Riemannian manifolds of dimension m and n respectively. A surjective C^∞-map π: M →B is a C^∞- submersion if it has maximal rank at any point of M, that is, each derivative map π_* of π is onto. For each b∈ B, π^{-1}(b) is submanifold of M of dimension (dim M - dimB), called the fibers of π.

Definition 2.2. ([11]). A vector field on M is called vertical if it is always tangent to the fiber and is called horizontal if it is always orthogonal to the fibers. For any p ∈M and let $\mathcal{V}_p$ = Ker(π_*) then, $\mathcal{V}_p = T_p\pi^{-1}$(b). The space $\mathcal{V}_p$ is called the vertical space at p and for every point p∈M the corresponding integrable distribution $\mathcal{V}$ is called the vertical distribution.

The sections of $\mathcal{V}$ are called the vertical vector fields and they determine a Lie subalgebra χ^v(M) of χ(M). The complementary distribution $\mathcal{H}$ of $\mathcal{V}$ is called the horizontal distribution. The sections of the horizontal distribution $\mathcal{H}$ are called the horizontal vector fields and they set up a subspace χ^h(M) of χ(M). Thus at any point p∈M, we have the orthogonal decomposition $T_pM = \mathcal{V}_p \oplus \mathcal{H}_p$.

The work on Riemannian submersions was initiated by O'Neill in [15] and he presented the foundation of Riemannian submersions as:

Definition 2.3.([11]). A C^∞- submersion $\pi: (M,g) \to (B,g')$ is called a Riemannian submersion if at each point p ∈M, π_{*p} preserves the length of the horizontal vectors, that is,

$$g(u,v) = g'_{\pi_p}(\pi_{*_p}u, \pi_{*_p}v),$$

where $u, v \in \mathcal{H}_p$, for p∈M.

O`Neill also defined the fundamental tensors of submersions T and A in [15] as below:

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathrm{vF} + \mathcal{V}\nabla_{\mathcal{V}\mathrm{E}}\mathcal{H}\mathrm{F}$$

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}\mathrm{F} + \mathcal{H}\nabla_{\mathcal{H}\mathrm{E}}\mathcal{V}\mathrm{F}.$$

where T acts as the second fundamental form of all the fibers and the tensor A determines the integrability of horizontal distributions.

Properties of the fundamental tensor T:

1. At each point, T_E is a skew-symmetric linear operator on the tangent space of M, and it reverses the horizontal and vertical subspaces.

2. T is vertical.

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3. For vertical vector fields T has symmetric property $T_V W = T_W V$.

Properties of the fundamental tensor A:

1. At each point, A_E is a skew-symmetric linear operator on the tangent space of M, and it reverses the horizontal and vertical subspaces.

2. A is horizontal.

3. For horizontal vector fields A has alteration property $A_X Y = A_Y X$.

There is one-to-one correspondence between the basic vector fields and their horizontal lifts and the following lemma shows the extent to which this correspondence preserves brackets, inner products and covariant derivatives.

Lemma 2.4.([15]). If X and Y are basic vector fields on M, then

1. $g(X,Y) = g(X_*, Y_*) \circ \pi$.

2. $\mathcal{H}[X, Y]$ is the basic vector field corresponding to $[X_*, Y_*]$.

3. $\mathcal{H}\nabla_X Y$ is the basic vector field corresponding to $\nabla^* X_*(Y_*)$.

The fundamental tensors T and A has relation with the various covariant derivatives given by the following lemma:

Lemma 2.5. ([15]). Let X and Y be horizontal vector fields, V and W be vertical vector fields. Then

1. $\nabla_V W = T_V W + \hat{\nabla}_V W$.

2. $\nabla_V X = T_V X + \mathcal{H}(\nabla_V X)$.

3. $\nabla_X V = A_X V + \mathcal{V}(\nabla_X V)$.

4. $\nabla_X Y = A_X Y + \mathcal{H}(\nabla_X Y)$.

Relation between various curvature tensors. ([15]) . For X, Y , Z, W horizontal vector fields and U, V , W , F the vertical vector fields O'Neill derived the following fundamental equations of submersion analogous to Gauss and Codazzi equations of an immersion.

1. $g(R_{UV}W, F) = g\big(\hat{R}_{UV}W, F\big) - g(T_U W, T_V F) + g(T_V W, T_U F)$.

2. $g(R_{UV}W, X) = g((\nabla_V T)_U W, X) - g((\nabla_U T)_V W, X)$.

3. $g(R_{XY}Z, H) = g(R^*_{XY}Z, H) - 2g(A_X Y, A_Z H) + g(A_Y Z, A_X H) + g(A_Z X, A_Y H)$.

4. $g(R_{XY}Z, V) = g((\nabla_Z A)_X Y, V) + g(A_X Y, T_V Z) - g(A_Y Z, T_V X) - g(A_Z X, T_V Y)$.

5. $g(R_{XV}Y, W) = g((\nabla_X T)_V W, X) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y) + g(A_X V, A_Y W)$.

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Corollary 2.6. ([15]). Let $\pi: M \to B$ be a submersion, and let K, K^* and $\widehat{K}$ be the sectional curvatures of M, B and the fibers respectively.If X, Y are the horizontal vector fields at a point of M and V and W are the vertical, then

1. $K(P_{VW}) = \widehat{K}(P_{VW}) - \frac{g(T_V V, T_W W) - \|T_V W\|^2}{\|V \wedge W\|^2}$.
2. $K(P_{XV}) \| X \|^2 \| V \|^2 = g((\nabla_X T)_V V, X) + \| A_X V \|^2 \| T_V X \|^2$.
3. $K(P_{XY}) = K_*\left(P_{X_* Y_*}\right) - \frac{3\|A_X Y\|^2}{\|X \wedge Y\|^2}$, where $X_* = \pi_*(X)$.

3. Screen lightlike submersion

Before giving the main results of screen lightlike submersion some brief introduction about lightlike manifolds is needed, for more details on this topic see [8].

Definition3.1.([8]). Let $(M; g)$ be a real n-dimensional smooth manifold, where g is a symmetric tensor field of type $(0; 2)$. The radical or null space of $T_x M$ is a subspace, denoted by $RadT_x M$ of $T_x M$ defined by $RadT_x M = \{\xi \in T_x M : g(\xi, X) = 0, X \in T_x M\}$ .The dimension of $RadT_x M$ is called the nullity degree of g. Suppose at each point $x \in M$ the mapping $RadTM$ assigns the subspace of $T_x M$ namely $RadT_x M$ of dimension $r > 0$, then $RadTM$ is called the radical distribution of M and the manifold M is called **lightlike manifold** if $0 < r \leq n$.

If in addition $RadTM$ is integrable and there exists a local coordinate system (x^α), $\alpha \in \{1,2, \ldots . r\}$ on a leaf L of $RadTM$ with $x^i = c^i, i \in \{r+1, r+2, \ldots \ldots . m\}$ and the matrix of g is

$$[g_{ij}] = \begin{pmatrix} 0_{r\ ,\ \ r} & 0_{r\ ,n-r} \\ 0_{n-r,\ \ r} & g_{ij}\,(x^1, \ldots, x^n) \end{pmatrix}$$

so that $\frac{\partial g_{ij}}{\partial x^\alpha} = 0, \forall\, i, j \in \{r+1, r+2, \ldots . m\}$, where $\left\{\frac{\partial}{\partial x^\alpha}\right\}, \alpha \in \{1,2, \ldots . r\}$ are the natural frame fields, then the lightlike manifold is called **Reinhart**. Considering $S(TM)$ as the complementary distribution to $RadTM$ in TM we get decomposition $TM = S(TM) \oplus Rad(TM)$, where $S(TM)$ is called the screen distribution.For Reinhart manifolds we have the following important result.

Theorem 3.2.([8]). Let (M, g) be a lightlike manifold. Then the following assertions are equivalent:

1. $(M; g)$ is a Reinhart lightlike manifold.
2. $RadTM$ is a killing distribution.
3. There exists a torsion free linear connection ∇ on M such that g is parallel tensor field with respect to ∇.

In case of Riemannian submersion splitting of tangent space of source manifoldinto two parts namely the vertical and the horizontal spaces plays a crucial role. In case of lightlike submanifold also the tangent space got decomposed into two parts one is called the radical distribution and other is called the screen distribution. On the basis of this similarity of

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splitting of tangent spaces Sahin in [18] introduced and studied a new submersion which can be treated as a lightlike version of Riemannian submersion, namely screen lightlike submersion between lightlike manifold and a semi-Riemannian manifold, as below:

Definition 3.3.([18]). Let $(M, g_M, S(TM))$ be an r-lightlike manifold and (N, g_N)a semi-Riemannian manifold. Then a smooth mapping $\varphi: (M, g_M, S(TM)) \longrightarrow (N, g_N)$ is a called a **screen lightlike submersion** if

1. at every $p \in M, \mathcal{V}_p = Ker(d\varphi)_p = Rad\ T_pM,$
2. at each point $p \in M$, the differential $d\varphi_p$ restrict to an isometry of the horizontal space $\mathcal{H}_p = S(TM)_p$ onto $T_{\varphi(p)}N.$

Sahin also gave the example of such submersion by taking $M = R^4_{2,1,1}$and $N = R^2_{0,1,1}$with the degenerate metric $g_1 = -(dx_3)^2 + (dx_4)^2$ and Lorentzian metric $g_2 = -(dy_1)^2 + (dy_2)^2$, respectively, with canonical coordinates x_1, x_2, x_3, x_4 and y_1, y_2 respectively and proved that the map $\varphi : R^4_{2,1,1} \longrightarrow R^2_{0,1,1}$ defined by $(x_1, x_2, x_3, x_4) \longrightarrow \left(\frac{2x_3+x_4}{\sqrt{3}}, \frac{x_3+2x_4}{\sqrt{3}}\right)$, is a screen lightlike submersion.

Lemma 3.4. ([18]). Let $\varphi: (M, g_M, S(TM)) \longrightarrow (N, g_N)$ be a screen lightlike submersion and X, Y basic vector fields of M. Then

1. $g_M(X, Y) = g_N\big(d\varphi(X), d\varphi(Y)\big) \circ \varphi.$
2. The horizontal part $[X, Y]^p$ of [X, Y] is a basic vector field and corresponds to $[\tilde{X}, \tilde{Y}]$.
3. For $\xi \in \Gamma(Rad\,TM)$, [X, ξ] $\in \Gamma(Rad(TM))$.

Lemma 3.5. ([18]). Let $\varphi: (M, g_M, S(TM)) \longrightarrow (N, g_N)$ be a screen lightlike submersion and M is Reinhart manifold with X , Y basic vector fields of M. Then $(\nabla_X Y)^P$ is the basic vector field corresponding to $\nabla^N_{\tilde{X}} \tilde{Y}$.

Lemma 3.6. ([18]). Let $\varphi: (M, g_M, S(TM)) \longrightarrow (N, g_N)$ be a screen lightlike submersion and M is Reinhart manifold and X , X_1 , X_2 , X_3 basic vector fields and $\xi \in \Gamma(RadTM)$. Then

$$R(X_1, X_2)X_3 = R^*(X_1, X_2)X_3 + A_{X_1}\nabla^*_{X_2}X_3 + Q\nabla_{X_1}A_{X_2}X_3 - A_{X_2}\nabla^*_{X_1}X_3 - Q\nabla_{X_2}A_{X_1}X_3 - A_{P[X_1,X_2]}X_3 - P\nabla_{Q[X_1,X_2]}X_3 - T_{Q[X_1,X_2]}X_3$$

and

$$R(X,\xi)\xi = Q\nabla_X\widehat{\nabla}_\xi\xi - \widehat{\nabla}_\xi(Q\nabla_X\xi) - \widehat{\nabla}_{[X,\xi]}\xi.$$

Theorem 3.7. ([18]). Let Let $\varphi: (M, g_M, S(TM)) \longrightarrow (N, g_N)$ be a screen lightlike submersion and M is Reinhart manifold. Let X_1, X_2 horizontal vector fields spanning 2-planes and $\xi \in \Gamma(\,Rad\,TM)$. Then we have

$$K_\xi(H) = 0$$

and

$$K(X_1, X_2) = K^N\big(\tilde{X}_1, \tilde{X}_2\big).$$

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Here K and K^N are sectional curvatures of M and N respectively.

4. Lightlike submersion

The lightlike submersion from a semi-Riemannian manifold onto lightlike manifold was given by Sahin and Gunduzalp in [19]. Following are the main results of their work:

Definition 4.1. ([19]). Let $f:(M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth submersion where (M_1, g_1) is semi-Riemannian manifold and (M_2, g_2) be an r-lightlike manifold. $Kerf_*$ and $(kerf_*)^{\perp}$ at a point p, is defined to be the sets

$$Kerf_* = \{X \in T_p(M_1): f_*(X) = 0\}$$

and

$$(kerf_*)^{\perp} = \{Y \in T_p(M_1): g_1(X,Y) = 0, \forall\, X \in Kerf_*\}.$$

As the space $T_p(M_1)$ is a semi-Riemannian $(kerf_*)^{\perp}$ is not necessarily be complementary to $Kerf_*$ and hence $Kerf_* \cap (kerf_*)^{\perp} \neq \{0\}$. Taking $\Delta = Kerf_* \cap (kerf_*)^{\perp}$, $Kerf_* = \Delta \perp S(Kerf_*)$ and $(kerf_*)^{\perp} = \Delta \perp S(kerf_*)^{\perp}$ where $S(Kerf_*)$ and $S(kerf_*)^{\perp}$ are complementary subspaces of Δ in $Kerf_*$ *and* $(kerf_*)^{\perp}$, respectively. Thus $T_p(M_1) = S(Kerf_*) \perp S(kerf_*)^{\perp}$ and $tr(Kerf_*) = ltr(Kerf_*) \perp S(kerf_*)^{\perp}$ ([8], proposition 2.4). For $\mathcal{V} = Kerf_*$, $\mathcal{H} = tr(Kerf_*)$ there are following four categories of lightlike submersions:

Case 1. If $0 < (dim\Delta = r) < \min\{\dim(Kerf_*), dim(kerf_*)^{\perp}\}$ and f_* preserves the length of horizontal vectors, that is, $g_1(X,Y) = g_2(f_*X, f_*Y)$, for $X, Y \in \Gamma(\mathcal{H})$ then f is called ***r*-lightlike submersion**.

Case 2. If $dim\Delta = \dim(Kerf_*) < dim(kerf_*)^{\perp}$, then $\mathcal{V} = \Delta$ and $\mathcal{H} = ltr(Kerf_*) \perp S(kerf_*)^{\perp}$ and f is called **isotropic-lightlike submersion**

Case 3. If $dim\Delta = dim(kerf_*)^{\perp} < \dim(Kerf_*)$, then $\mathcal{V} = \Delta \perp S(Ker\, f_*)$ and $\mathcal{H} = ltr(Kerf_*)$ then f is called **co-isotropic lightlike submersion.**

Case 4. If $dim\Delta = \dim(Kerf_*) = dim(kerf_*)^{\perp}$, then $\mathcal{V} = \Delta$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called **totallylightlike submersion**.

In [19], Sahin presented example of each of above type of lightlike submersion. The following is the example of 1- lightlike submersion:

Taking $M_1 = R^4_{0,1,3}$ and $M_2 = R^2_{1,0,1,}$ with the Lorentzian metric $g_1 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2$ and degenerate metric $g_2 = (dy_2)^2$ respectively, with canonical coordinates x_1, x_2, x_3, x_4 and y_1, y_2 respectively and proved that the map $f : R^4_{0,1,3} \longrightarrow R^2_{1,0,1,}$ defined by $(x_1, x_2, x_3, x_4) \longrightarrow \left(x_1 + x_3\,,\, \frac{x_2 + 2x_4}{\sqrt{2}}\right)$ is 1- lightlike submersion.

Theorem 4.2. ([19]). Let $f:(M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth lightlike submersion. Then

1. If f is an r-lightlike or isotropic submersion , then M_2 is an r-lightlike manifold.
2. If f is co-isotropic or totally lightlike submersion, then M_2 is totally lightlike manifold.

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Definition 4.3. ([19]). Let $f:(M_1,g_1)\longrightarrow(M_2,g_2)$ be a smooth lightlike submersion and X,Y be arbitrary vector fields on M_1. Let $\hbar:TM_1\longrightarrow\mathcal{H}$ and $v:TM_1\longrightarrow\mathcal{V}$ denote the natural projections for the decomposition $TM_1=\mathcal{V}\oplus\mathcal{H}$. Let ∇ be the Levi-Civita connection of (M_1,g_1). Then the fundamental tensor of lightlike submersion were defined by T and A using the following formulas:

$$T_XY=\hbar\,\nabla_{vX}\nu Y+\nu\nabla_{\nu X}\hbar Y$$

$$A_XY=v\nabla_{\hbar X}\hbar Y+\hbar\nabla_{\hbar X}\nu Y.$$

Lemma4.4. ([19]). Let $f:(M_1,g_1)\longrightarrow(M_2,g_2)$ be a smooth lightlike submersion. If X,Y are the basic vector fields on M_1, then

1. $g_1(X,Y)=g_2(\tilde{X},\tilde{Y})\circ f$.
2. $\hbar[X,Y]$ is the basic vector field corresponding to $[\tilde{X},\tilde{Y}]$.

In Riemannian submersion,$\hbar\nabla_XY$ is the basic vector field corresponding to $\nabla^{M_2}_{\tilde{X}}\tilde{Y}$, where ∇^{M_2} is the linear connection of, M_2, but the following theorem shows that in lightlike submersion this property is true as a particular case only.

Theorem4.5. ([19]). Let $f:(M_1,g_1)\longrightarrow(M_2,g_2)$ be a smooth lightlike submersion, where M_1ia semi-Riemannian and M_2 is a Reinhart lightlike manifold. Then for the basic vector fields X,Y,$\hbar\nabla_XY$ is the basic vector field corresponding to $\nabla^{M_2}_{\tilde{X}}\tilde{Y}$.

Theorem 4.6. ([19]). Let $f:(M_1,g_1)\longrightarrow(M_2,g_2)$ be an r- lightlike submersion or an isotropic submersion, where M_1ia semi-Riemannian and M_2 is a Reinhart lightlike manifold. Then for any $X\in\Gamma(ltr(Kerf_*))$ and $U,V,W,F\in\Gamma(\Delta)$ we have

$$g_1(R(U,V)W,X)=g_1\big(\hat{R}(U,V)W,X\big)+g_1(T_UT_VW,X)-g_1(T_VT_UW,X),$$

$$g_1(R(U,V)W,F)=g_1((\nabla_UT)_VW,F)-g_1((\nabla_VT)_UW,F),$$

where ∇ is the Levi-Civita connection on M_1, R and $\hat{R}$ are the Riemannian curvature tensor field of M_1 and fiber, respectively.

Theorem4.7. ([19]). Let $f:(M_1,g_1)\longrightarrow(M_2,g_2)$ be an r- lightlike submersion or an isotropic submersion, where M_1 is a semi-Riemannian and M_2 is a Reinhart lightlike manifold.Then

$$K_{M_2}\big(\tilde{Z},\tilde{U}\big)=K_{M_1}(Z,U)-g_1(A_ZA_UU,Z)+g_1(A_UA_ZU,Z)-g_1(U,T_{v[Z,U]}Z),$$

for any $Z\in\Gamma(S(Kerf)^{\perp}_{*})$ and $U\in\Gamma(ltr(kerf_*))$, where K_{M_2} is the null sectional curvature of M_2 and K_{M_1} is the null sectional curvature of M_1.

5. Screen conformal lightlike submersion

As a generalization of Riemannian submersions the concept of horizontally conformal submersion was introduced in which the second condition of Riemannian submersion changes to more general condition, that is, there exists a function $\lambda: M\longrightarrow R^+$ such that

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$$g_B\big(d\varphi(X), d\varphi(Y)\big) = \lambda^2 g_M(X,Y), \qquad \forall\, X, Y \in \Gamma(TM).$$

In 2007, Sahin [17] studied screen conformal submersion between lightlike manifolds and semi-Riemannian manifolds, which can be considered as a lightlike version of horizontally conformal submersions. Sahin defined O'Neill tensors for this submersion and proved that these tensors have some properties different than that of O'Neill tensors in case of Riemannian submersion. Sahin also defined radical and screen homothetic submersions as a special case of screen conformal submersion. He proved that the source manifold M is Reinhart if the submersion is radical homothetic. The curvature relations between base manifold and the total manifold were obtained. The concept and the conditions for the harmonicity for these maps were also studied. The main results of their study are as below.

Definition 5.1. ([17]). Let (M,g) and (N,g_N) be a lightlike and semi-Riemannian manifold respectively with a submersion $\phi : M \longrightarrow \bar{M}$. The map ϕ is said to be **screen conformal lightlike submersion** if the following conditions are satisfied:

1. $Kerd\phi = RadTM$.

2. There exists a function $\lambda : M \longrightarrow R^+$ such that , for $x \in M$, $g_N\big(d\phi(X), d\phi(Y)\big) = \lambda^2(x) g(X,Y)$, $\forall\ X\,,\ Y \in \Gamma\big(S(TM)\big)$.

Sahin also gave an example of such submersion by taking $M = R^4_{2,1,1}$and $N = \tilde{R}^2_{0,1,1}$with the degenerate metric $g_1 = -(dx_3)^2 + (dx_4)^2$ and Lorentzian metric $g_2 = -(dy_1)^2 + (dy_2)^2$ respectively, with canonical coordinates x_1, x_2, x_3, x_4 and $y_1\,, y_2$ respectively and proved that the map $\varphi : R^4_{2,1,1} \longrightarrow \tilde{R}^2_{0,1,1}$ defined by $(x_1, x_2, x_3, x_4) \longrightarrow (sinh x_3,\ cosh x_4,\ cosh x_3,\ sinh x_4)$, is screen conformal lightlike submersion.

Lemma5.2. ([17]). Let $\phi : M \longrightarrow N,$ be screen conformal lightlike submersion and X, Y be basic vector fields of M. Then

1. The horizontal part $[X,Y]^P$ of $[X,Y]$ is the basic vector field and corresponds to $[\tilde{X}, \tilde{Y}]$.

2. For $\xi \in \Gamma(RadTM), [X,\xi] \in \Gamma(RadTM)$.

Lemma5.3. ([17]). Let $\phi : (M,g) \longrightarrow (N,g_N),$ be screen conformal lightlike submersion and $X, Y \in S(TM)$. Then $RadTM$ is shear free if $\xi g_N(d\phi(X), d\phi(Y)) = 0, \forall\ \xi \in \Gamma(RadTM)$.

Using above lemma, the Theorem 3.2 of the Section 2 and from the definition of radical homothetic map, Sahin proved the following result.

Theorem5.4. ([17]). Let $\phi : (M,g) \longrightarrow (N,g_N)$ be radical homothetic lightlike submersion Then the manifold M is Reinhart if $\xi g_N(d\phi(X), d\phi(Y))$, where $X, Y \in S(TM)$ and $\xi \in \Gamma(RadTM)$.

Theorem5.5 ([17]). Let $\phi : (M,g) \longrightarrow (N,g_N),$ be a screen conformal lightlike submersion, X_1 , X_2 be basic vector fields and $\xi \in \Gamma(Rad(TM))$. Then

$$K_\xi(H) = 0, \qquad K(X_1\,, X_2) = \frac{1}{\lambda^2} K^*(\bar{X}_1\,,\ \bar{X}_2),$$

Here K, K^* are sectional curvatures of $M\ and\ N$ respectively.

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6. Harmonic maps from lightlike manifolds

Harmonic maps are closely related to holomorphic maps in several complex variables, to non-linear field theory in theoretical physics, to the theory of stochastic processes and to the theory of liquid crystals in material sciences. The concept of Harmonic maps has been developed in last sixty years and is still an active area of research in differential geometry of mathematical physics. The concept of harmonic maps also constitutes a significant tool for both global analysis and differential geometry. In 1964, J. Eells and J. H. Sampson [9] worked on harmonic mappings of Riemannian manifolds.

Definition 6.1. ([17]). Let $\phi:(M,g)\longrightarrow(N,g_N)$ be a smooth mapping between the Riemannian manifolds. The differential $d\phi$ of ϕ can be viewed as a section of the vector bundle $T^*M\otimes\phi^{-1}TN=Hom(TM,\phi^{-1}TN)$ over manifold M. This bundle has a connection ∇ induced from the Levi-Civita connection ∇^M of M and the pull back connection ∇^ϕ of $\phi^{-1}TN$. Then the second fundamental form of ϕ is given by

$$\nabla d\phi(X,Y)=\nabla_X^\phi\big(d\phi(Y)\big)-d\phi(\nabla_X^M Y),$$

for any $X,Y\in\Gamma(TM)$ and is always symmetric. A differential map f between Riemannian manifolds is called harmonic if $tr(\nabla df)=0$. (for more details on harmonic maps see[2,9]) But in case M is lightlike trace of second fundamental form is meaningless on the radical part. To deal with this situation Sahin in [17], presented the following work for harmonic maps on lightlike manifolds.

Definition 6.2. ([17]). Let (M,g) be a lightlike manifold and (N,g_N)a semi-Riemannian manifold. A smooth map $\phi:(M,g)\longrightarrow(N,g_N)$ is called harmonic if

1. $\nabla d\phi=0\ on\ RadTM.$
2. $tr|_{S(TM)}\nabla d\phi=0.$

Lemma 6.3. ([17]). Let $\phi:(M,g)\longrightarrow(N,g_N)$ be a screen conformal lightlike submersion, where M is Reinhart lightlike manifold. Then

$$\nabla d\phi(\xi_1,\xi_2)=0$$

for $\xi_1,\xi_2\in\Gamma\big(S(TM)\big)$ and

$$\nabla d\phi(X,Y)=X(ln\lambda)d\phi(Y)+Y(ln\lambda)d\phi(X)-d\phi(grad\ ln\lambda)g_{S(TM)}(X,Y)$$

for X , Y $\in\Gamma(S(TM))$.

Using the above Lemma, Sahin proved the following theorem which gives necessary and sufficient condition for a screen conformal submersion to be lightlike harmonic.

Theorem6.4. ([17]). Let $\phi:(M,g)\longrightarrow(N,g_N)$ be a screen conformal lightlike submersion and $\dim\big(S(TM)\big)=n$. Then

1. If $n=2$, ϕ is a lightlike harmonic map.
2. If $n\neq 2$, then ϕ is a lightlike harmonic if and only if ϕ is screen homothetic.

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