Characterization of Growth Parameters of Solutions of Poisson's Equation in Three Variables

[1]Devendra Kumar, [2]Vandna Jain

[1]*Department of Mathematics [Research and Post Graduate Studies], M.M.H. College, Model Town, Ghaziabad-201001,U.P.,India. E-Mail:d_kumar001@rediffmail.com*

[2]*Research Scholar, Department of Applied Sciences ,I K Gujral PTU Kapurthala,(Punjab),India. E-Mail:vandnajain.mittal@gmail.com*

Abstract. In this paper the Poisson's equation in three variables has been considered and attempt has been done to study the growth properties of solutions of this equation by using function theoretic method. Moreover, we have characterized the order and type of an associated analytic function $F \in C^3$ in terms of coefficients occurring in its power series expansion which have not been studied so far. Our results apply satisfactory for time dependent problems in R^3.

Keywords and Phrases. Poisson's equation, Whittaker-Bergman operator, generalized axisymmetric potentials, harmonic functions, function theoretic method, generalized growth.

1 Introduction

The solution of the partial differential equations

$$L^{(K)}[\emptyset] \equiv \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{K}{y}\frac{\partial \emptyset}{\partial y} = 0, K \geq 0 \tag{1.1}$$

are called the generalized axisymmetric potentials (GASP). For K=n-2, n≥2, with $x=x_1$, $y=(x_2^2+x_3^2+\ldots+x_n^2)^{1/2}$ in some neighborhood $\Omega \subset E^n$ of the origin where GASP's are subject to the initial data.

$$\emptyset \in C^2(\Omega), \text{ satisfies (1.1) for all } (x,y) \in \Omega, y \neq 0, \frac{\partial \emptyset(x,0)}{\partial y} = 0 \tag{1.2}$$

on the intersection of Ω with x-axis has been considered by Kumar and Arora [7] and studied the growth parameters order and type of entire function GASP in terms of polynomial approximation errors by using the function theoretic approach. Such functions are necessarily symmetric i.e., satisfy $\emptyset(x,y) = \emptyset(x,-y)$. We say that $\emptyset$ is regular in some region $\Omega' \supset \mathrm{cl}(\Omega)$. In this case the GASP's $\emptyset$ are harmonic in R^n, i.e., satisfy Laplace's equation

$$\frac{\partial^2 \emptyset}{\partial {x_1}^2} + \frac{\partial^2 \emptyset}{\partial {x_2}^2} + \cdots + \frac{\partial^2 \emptyset}{\partial {x_n}^2} = 0.$$

For each $\emptyset$ with domain Ω, there is a unique associated function f,

$$f(\eta) = \sum_{k=0}^{\infty} a_k \eta^k,$$

analytic on the corresponding axiconvex domain (if x+iy$\in \Omega$ implies x+$i\beta$y$\in \Omega$ for every $\beta \in [-1,1]$) whose A_n associate is defined by

$$\emptyset(z) = A_n(f) = \alpha_n \int_L f(\sigma)(\zeta - \zeta^{-1})^{n-3} \frac{d\zeta}{\zeta} \quad , \ \sigma = x + \frac{iy}{2}(\zeta + \zeta^{-1}), \alpha_n = \frac{\Gamma(n-2)}{(4i)^{n-2}\Gamma\left(\frac{n}{2}-1\right)},$$

$$L \equiv \{\zeta = e^{it} : 0 \le t \le \pi\}, |z - z^0| < \epsilon, \quad z = x + iy, \quad z^0 = x^0 + iy^0.$$

It is an initial point of definition for $\emptyset(z), \epsilon > 0$ is sufficiently small, and integration path is the upper semi-circular arc connecting +1 to -1. It should be noted that the A_n associate of the GASP is the analytic continuation of its restriction to the axis of symmetry, i.e.,

$f(z) = \emptyset(z, 0)$.

In this paper we shall investigate the growth properties of solutions to the Poisson's equation in three variables, namely

$$\Delta u \equiv \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} = \delta(x_1, x_2, x_3). \tag{1.3}$$

We may generate solutions to the equation (1.3) by means of an integral representation similar to Whittaker-Bergman operator which maps functions of two complex variables into solutions of Laplace's equation, $\Delta H(\mathbb{X}) = 0$.

$$H(\mathbb{X}) = K_3(f) \equiv \frac{1}{2\pi i} \int_{\mathscr{L}} f(t, \zeta) \frac{d\zeta}{\zeta} \quad , \tag{1.4}$$

$$t = \left[-(x_1 - ix_2)\frac{\zeta}{2} + x_3 + (x_1 + ix_2)\frac{1}{2\zeta}\right], \|\mathbb{X} - \mathbb{X}^0\| < \epsilon, \mathbb{X} \equiv (x_1, x_2, x_3),$$

$\mathbb{X}^0 \equiv ({x_1}^0, {x_2}^0, {x_3}^0)$,

where $\mathscr{L}$ is closed differentiable arc in ζ – plane , and $\epsilon > 0$ is sufficiently small.

Now we introduce the integral representation ${K_3}^*(F)$,

$$u(\mathfrak{X}) = K_3{}^{*}(F) \equiv \frac{1}{2\pi i}\int_{\mathscr{L}} F(t, t^{*}, \zeta)\frac{d\zeta}{\zeta}\ , \tag{1.5}$$

where F is an analytic function of the three complex variables $t, t^{*}, \zeta; t$ is the same as above,

$$t^{*} = \left[(x_1 - ix_2)\frac{\zeta}{2} + x_3 - (x_1 + ix_2)\frac{1}{2\zeta}\right], \|\mathfrak{X} - \mathfrak{X}^0\| < \epsilon.$$

It should be noticed that, for $|\zeta| = 1$ *and* $\mathfrak{X} \equiv (x_1, x_2, x_3)$ a real point, then $t^{*} = \bar{t}$.If $\frac{\partial F}{\partial t} = 0$, or $\frac{\partial F}{\partial t^{*}} = 0$,then F is a function of t *or* t^{*} respectively; in these cases $K_3{}^{*}(F)$ becomes the Whittaker-Bergman operator, and u($\mathfrak{X}$) is a harmonic function. If $\delta(\mathfrak{X}) \in C^2$, then we have by interchanging orders of differentiation and integration

$$\delta(\mathfrak{X}) = \Delta u = \frac{1}{2\pi i}\int_{|\zeta| = 1} \Delta F \frac{d\zeta}{\zeta} = \frac{1}{4\pi i}\int_{|\zeta| = 1} (\zeta - \zeta^{-1})^2 \frac{\partial^2 F}{\partial t \partial t^{*}}\frac{d\zeta}{\zeta} \tag{1.6}$$

Hence we may obtain a particular solution of $\Delta u = \delta$, with the representation (1.5) if F is a solution of integral equation (1.6). In order, to see how $K_3{}^{*}(F)$ transforms analytic functions of three variables into solutions of (1.3), we consider the integrals

$$u_{n,m,\lambda}(\mathfrak{X}) = K_3{}^{*}\left(t^n t^{*m} \zeta^{\lambda}\right) = \frac{1}{2\pi i}\int_{|\zeta| = 1} t^n t^{*m} \zeta^{\lambda}\frac{d\zeta}{\zeta}\ , \qquad -n - m \leq \lambda \leq n + m,$$

$$= \frac{1}{2\pi i} r^{n+m} n!\, m! \int_{|\zeta| = 1} \left(\sum_{\nu=-n}^{n} P_n{}^{\nu}(\xi)\frac{i^{-\nu}\zeta^{-\nu}}{(n+\nu)!}e^{i\nu\vartheta}\right)\left(\sum_{\mu=-m}^{m}(-1)^{\mu} P_m{}^{\mu}(\xi)\frac{i^{-\mu}\zeta^{-\mu}}{(m+\mu)!}e^{i\mu\vartheta}\right)\zeta^{\lambda}\frac{d\zeta}{\zeta}$$

$$= e^{i\lambda\vartheta} i^{-\lambda} r^{n+m} n!\, m! \sum_{\nu \in N(m,n)} \frac{P_n{}^{\nu}(\xi) P_m{}^{\lambda-\nu}(\xi)}{(n+\nu)!(m+\lambda-\nu)!} \tag{1.7}$$

where N(m,n) is the set of indices $\{\nu\colon |\nu| \leq n, |\lambda - \nu| \leq m\}$.

Consequently, a function which is regular in a neighborhood of origin can be expanded as a compactly convergent series

$$u(\bar{\mathfrak{X}}) = \sum_{n,m=0}^{\infty} \sum_{\lambda=-n-m}^{n+m} a_{nm\lambda} u_{nm\lambda}\,(\bar{\mathfrak{X}}) \tag{1.8}$$

in the sphere $S(R)\colon |\bar{\mathfrak{X}}| < R$ whose radius is the distance from the origin to the nearest singularity.

Let D be a simply connected domain about the origin in R^4. A harmonic function $u \in C^2(D)$ is a solution of Laplace's equation [3, p.494]

$$\left[\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}+\frac{4}{r^2}\left[\frac{\partial^2}{\partial\theta^2}+cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\theta^2}-2\cos\theta\frac{\partial^2}{\partial\phi\partial\theta}\right)\right]\right)\right]u(\overline{\mathrm{X}})=0. \quad (1.9)$$

Each harmonic function u is associated with a unique analytic function

$$F(t,t^*,\zeta)=\sum_{n,m=0}^{\infty}\sum_{\lambda=-n-m}^{n+m}a_{nm\lambda}\,t^n{t^*}^m\zeta^\lambda, \quad (1.8)$$

of three complex variables. The initial domain of definition of $F \in C^3$ is

$$D_{\epsilon,R}: |t| < R, |t^*| < R, 1-\epsilon < |\zeta| < 1+\epsilon,$$

for sufficiently small positive ϵ.

The above facts are summarized in

Theorem A. For each function u that is harmonic at origin in sphere $S(R): |\overline{\mathrm{X}}| < R$ there is a unique associated function F of three complex variables analytic in the disc $D_{\epsilon,R}$ and conversely.

The theory in R^3 is well developed. Several authors studied boundary value problems[3], value distribution and growth properties ([1], [2], [5], [6], [8], [9], [11]) of harmonic functions in R^3. Sometimes it is reasonable and become interesting when we restrict the time dependent problems in R^3, it leads to the study of harmonic functions in R^4. Since there is an isometry between analytic functions of three complex variables and harmonic functions in R^4 on suitable domain of definition [10] , we have attempted to study the growth properties of function F analytic in three complex variables and using the Laplacian type integral operator (and inverse). We can study the growth properties of harmonic function $u \in R^4$ i.e., the solution of equation (1.3) which has not been studied so far. Moreover, in this paper we characterize the order and type of $F \in C^3$ in terms of coefficients $a_{nm\lambda}$ occurring in its power series expansion (1.10).

2 Some Definitions

The maximum modulus of $F \in C^3$ is defined as in complex function theory

$$M(r_1,r_2,r_3,F)=\max_{(|t|=r_1,|t^*|=r_2,|\zeta|=r_3)} F(t,t^*,\zeta),$$

$t < R, t^* < R, 1-\epsilon < |\zeta| < 1+\epsilon.$

The growth of a function F, analytic in $D_{\epsilon,R}$ as determined by its maximum modulus function $M(r_1, r_2, r_3, F)$ can be studied in several different ways. To measure the growth of F with respect to all the variables simultaneously, the concept of $D_{\epsilon,R}$ –order and $D_{\epsilon,R}$ –type introduced by Juneja and Kapoor[4] has been used.

Let

$$M_{D_{\epsilon,R}}(\theta, F) = \max_{r_1, r_2, r_3 \in \theta D_{\epsilon,R}} M(r_1, r_2, r_3, F), 0 < \theta < 1.$$

We define the $D_{\epsilon,R}$ –order $\rho_{D_{\epsilon,R}}(F)$ of F as

$$\rho_{D_{\epsilon,R}}(F) = \limsup_{\theta \to 1} \left\{ \frac{log^+ log^+ M_{D_{\epsilon,R}}(\theta,F)}{-\log(1-\theta)} \right\} \tag{2.1}$$

if $0 < \rho_{D_{\epsilon,R}}(F) < \infty$, the $D_{\epsilon,R}$ –type $T_{D_{\epsilon,R}}(F)$ of F is defined as

$$T_{D_{\epsilon,R}}(F) = \limsup_{\theta \to 1} \left\{ \frac{log^+ M_{D_{\epsilon,R}}(\theta,F)}{(1-\theta)^{-\rho_{D_{\epsilon,R}}(F)}} \right\} \tag{2.2}$$

3 Main Results

Now we prove

Theorem 3.1. Let $F(t, t^*, \zeta) = \sum_{n,m=0}^{\infty} \sum_{\lambda=-n-m}^{n+m} a_{nm\lambda}\, t^n t^{*^m} \zeta^\lambda$ be analytic in the polydisc $D_{\epsilon,R}$ and has $D_{\epsilon,R}$ –order $\rho_{D_{\epsilon,R}}(F), 0 \le \rho_{D_{\epsilon,R}}(F) \le \infty$. Then

$$\frac{\rho_{D_{\epsilon,R}}(F)}{\rho_{D_{\epsilon,R}}(F)+1} = \limsup_{n+m+\lambda \to \infty} \left\{ \frac{log^+ log^+ a_{nm\lambda} {r_1}^n {r_2}^m {r_3}^\lambda}{\log(n+m+\lambda)} \right\} \tag{3.1}$$

the left hand side is interpreted as 1 if $\rho_{D_{\epsilon,R}}(F) = \infty$.

Proof. Consider the function $\beta(t, t^*, \zeta, \omega)$ of four complex variables t, t^*, ζ, ω defined by

$$\beta(t, t^*, \zeta, \omega) = F\big(\omega(t, t^*, \zeta)\big) = \sum_{n,m=0}^{\infty} \sum_{\lambda=-n-m}^{n+m} a_{nm\lambda}\, \omega^{n+m+\lambda} t^n t^{*^m} \zeta^\lambda,$$

where $|\omega| < R$. Set

$$P_r(t,t^*,\zeta)=\sum_{n,m=0}^{\infty}\sum_{\lambda=-n-m}^{n+m}a_{nm\lambda}t^n{t^*}^m\zeta^\lambda,\quad n+m+\lambda=\gamma.$$

Then

$$\beta(t,t^*,\zeta,\omega)=\sum_{\gamma=0}^{\infty}P_r(t,t^*,\zeta)\,\omega^\gamma$$

is an analytic function of ω in finite disc of radius R. In view of Cauchy inequality for the coefficients of a power series of one variable, we get, for $0<\theta<1$ and $(t,t^*,\zeta)\in D_{\epsilon,R}$,

$$|P_r(t,t^*,\zeta)|\leq \max_{|\omega|=\theta R}\left\{\frac{|\beta(t,t^*,\zeta,\omega)|}{\theta^\gamma {r_1}^n {r_2}^m {r_3}^\lambda}\right\}. \tag{3.2}$$

Now

$$\max_{|\omega|=\theta R}|\beta(t,t^*,\zeta,\omega)|\leq \max_{|\omega|=\theta R}\max_{(|t|=r_1,|t^*|=r_2,|\zeta|=r_3)}\left|F\left(\omega(t,t^*,\zeta)\right)\right|$$

$$=\max_{(|\vartheta_1|=\theta r_1,|\vartheta_2|=\theta r_2,|\vartheta_3|=\theta r_3)}|F(\vartheta_1,\vartheta_2,\vartheta_3)|$$

$$\leq \max_{{r_1}^*,{r_2}^*,{r_3}^*\in\theta D_{\epsilon,R}} M({r_1}^*,{r_2}^*,{r_3}^*,F) = M_{D_{\epsilon,R}}(\theta,F)$$

Therefore we have

$$|P_r(t,t^*,\zeta)|\leq\frac{M_{D_{\epsilon,R}}(\theta,F)}{\theta^\gamma {r_1}^n {r_2}^m {r_3}^\lambda} \tag{3.3}$$

or

$$M_{D_{\epsilon,R}}(R,P_r)=\max_{r_1,r_2,r_3\in\theta D_{\epsilon,R}} M(r_1,r_2,r_3,P_r)$$

$$=\max_{r_1,r_2,r_3\in tD_{\epsilon,R}}\max_{(|t|=r_1,|t^*|=r_2,|\zeta|=r_3)}|P_r(t,t^*,\zeta)|$$

$$=\frac{M_{D_{\epsilon,R}}(\theta,F)}{\theta^\gamma {r_1}^n {r_2}^m {r_3}^\lambda}.$$

For every positive integer γ and for all θ, $0<\theta<1$, we have

$$M_{D_{\epsilon,R}}(R,P_r) \leq \frac{M_{D_{\epsilon,R}}(\theta,F)}{\theta^{\gamma} {r_1}^n {r_2}^m {r_3}^{\lambda}} \,. \tag{3.4}$$

Now let $\rho(F) \equiv \rho_{D_{\epsilon,R}}(F) < \infty$ and in view of definition of $\rho(F)$ for any $> 0, 0 < \theta < 1$, we get

$$log^+ M_{D_{\epsilon,R}}(\theta,F) < (1-\theta)^{-\rho(F)-\epsilon}.$$

Using (3.4), it follows that

$$M_{D_{\epsilon,R}}(R,P_r) < exp\{(1-\theta)^{-\rho(F)-\epsilon}\}\theta^{-\gamma} {r_1}^{-n} {r_2}^{-m} {r_3}^{-\lambda} \quad . \tag{3.5}$$

Now for $(r_1,r_2,r_3) \in \theta D_{\epsilon,R}$, we have

$$a_{nm\lambda} \leq \frac{M(r_1,r_2,r_3,P_r)}{{r_1}^n {r_2}^m {r_3}^{\lambda}}.$$

Minimizing the right hand side of above inequality for all $(r_1,r_2,r_3) \in D_{\epsilon,R}$, we obtain

$$a_{nm\lambda} \leq \; M_{D_{\epsilon,R}}(R,P_r).$$

Combining this inequality with (3.5), we have

$$a_{nm\lambda}\, {r_1}^n {r_2}^m {r_3}^{\lambda} \leq exp\{(1-\theta)^{-\rho(F)-\epsilon}\}\theta^{-\gamma} \,. \tag{3.6}$$

Minimizing the right hand side of (3.6), we get

$$a_{nm\lambda}\, {r_1}^n {r_2}^m {r_3}^{\lambda} < exp\left\{(1+\rho(F)+\epsilon)\left(\frac{n+m+\lambda}{\rho(F)+\epsilon}\right)^{(\rho(F)+\epsilon)/(\rho(F)+1+\epsilon)}\right\}$$

or $\displaystyle \frac{log^+ log^+ (a_{nm\lambda} {r_1}^n {r_2}^m {r_3}^{\lambda})}{\log\,(n+m+\lambda)} \leq \frac{(\rho(F)+\epsilon)}{(\rho(F)+1+\epsilon)} + O(1),$

or $$\limsup_{n+m+\lambda \to \infty} \left\{\frac{log^+ log^+ (a_{nm\lambda} {r_1}^n {r_2}^m {r_3}^{\lambda})}{\log\,(n+m+\lambda)}\right\} \leq \frac{\rho(F)}{\rho(F)+1} \quad . \tag{3.7}$$

If $\rho(F) = \infty$, we produced with an arbitrary large number in place of $\rho(F)+1$ and get 1 on the right hand side of (3.7).

In order to prove reverse inequality, let

$$\limsup_{n+m+\lambda\to\infty}\left\{\frac{\log^{+}\log^{+}(a_{nm\lambda}{r_1}^{n}{r_2}^{m}{r_3}^{\lambda})}{\log(n+m+\lambda)}\right\}=\alpha^{*}.$$

For any $\epsilon > 0$, there exists a non negative integer $\gamma(\epsilon)$ such that, for $+m+\lambda \geq \gamma(\epsilon)$,

$a_{nm\lambda}\, {r_1}^{n}{r_2}^{m}{r_3}^{\lambda} < exp\{(n+m+\lambda)^{\alpha^*+\epsilon}\}$.

We have for $0 < \theta < 1$,

$$M_{D_{\epsilon,R}}(\theta, F) \leq \max_{(r_1,r_2,r_3)\in D_{\epsilon,R}} \sum_{n+m+\lambda=0}^{\infty} a_{nm\lambda}\, {r_1}^{n}{r_2}^{m}{r_3}^{\lambda}\theta^{(n+m+\lambda)} \qquad (3.8)$$

$$\leq \sum_{n+m+\lambda\leq\gamma(\epsilon)} a_{nm\lambda}{r_1}^{n}{r_2}^{m}{r_3}^{\lambda}\theta^{(n+m+\lambda)} + \sum_{n+m+\lambda>\gamma(\epsilon)} \theta^{(n+m+\lambda)} exp(n+m+\lambda)^{(\alpha^*+\epsilon)}$$

$$< c_1\theta^{\gamma(\epsilon)} + c_2 + \sum_{\gamma=0}^{\infty}\theta^{\gamma}(1+\gamma)^3 \exp{(\gamma)}^{(\alpha^*+\epsilon)}.$$

Where c_1 and c_2 are constants. Now it is easy to see that

$$F^*(t,t^*,\zeta,\theta) = \sum_{n+m+\lambda=0}^{\infty} (1+n+m+\lambda)^3 exp(n+m+\lambda)^{(\alpha^*+\epsilon)} t^n {t^*}^m \zeta^{\lambda}\, \theta^{(n+m+\lambda)}$$

is analytic in finite disc $D_{\epsilon,R}$. The order of F^* is $(\alpha^*+\epsilon)/(1-\alpha^*-\epsilon)$.

Thus, using the definition of order in three complex variables any $\epsilon^{/} > 0$ and θ sufficiently close to 1,

$$M_{D_{\epsilon,R}}(\theta, F^*) < exp\left\{(1-\theta)^{-\left((\alpha^*+\epsilon)/(1-\alpha^*-\epsilon)\right)+\epsilon^{/}}\right\}.$$

In view of (3.8), we get

$$M_{D_{\epsilon,R}}(\theta, F) < c_1\theta^{\gamma(\epsilon)} + c_2 + M_{D_{\epsilon,R}}(\theta, F^*).$$

For θ sufficiently close to 1, we have

$$M_{D_{\epsilon,R}}(\theta, F) < c_1\theta^{\gamma(\epsilon)} + c_2 + \ exp\left\{(1-\theta)^{-\left((\alpha^*+\epsilon)/(1-\alpha^*-\epsilon)\right)+\epsilon^{/}}\right\}.$$

Since ϵ and $\epsilon^{/}$ are arbitrary, the above inequality implies that

$$\rho(F) \leq \frac{\alpha^*}{(1-\alpha^*)}$$

or

$$\frac{\rho(F)}{\rho(F)+1} \leq \alpha^* \ . \tag{3.9}$$

Combining (3.7) and (3.9), the proof is completed.

Theorem 3.2. Let $F(t,t^*,\zeta) = \sum_{n,m=0}^{\infty}\sum_{\lambda=-n-m}^{n+m} a_{nm\lambda}\, t^n {t^*}^m \zeta^\lambda$ be analytic in the polydisc $D_{\epsilon,R}$ and has order $\rho(F), (0 < \rho(F) < \infty)$ and type $T_{D_{\epsilon,R}}(F), \left(0 \leq T_{D_{\epsilon,R}} \leq \infty\right)$, then

$$\frac{(\rho(F)+1)^{\rho(F)+1}}{(\rho(F))^{\rho(F)}} T_{D_{\epsilon,R}}(F) = \limsup_{n+m+\lambda\to\infty} \left\{\frac{(\log^+ a_{nm\lambda} {r_1}^n {r_2}^m {r_3}^\lambda)^{\rho(F)+1}}{(n+m+\lambda)^{\rho(F)}}\right\} \tag{3.10}$$

Proof. Bearing in mind the technique employed in the proof of Theorem 3.1, we have

$$a_{nm\lambda} {r_1}^n {r_2}^m {r_3}^\lambda \leq exp\left\{(T(F)+\epsilon)(1-\theta)^{-\rho(F)}\theta^{-\gamma}\right\}, \tag{3.11}$$

$$T(F) \equiv T_{D_{\epsilon,R}}(F).$$

Minimizing the right hand side of (3.11), we get

$$a_{nm\lambda} {r_1}^n {r_2}^m {r_3}^\lambda < exp\left\{\left(\frac{T(F)+\epsilon}{\rho(F)}\right)^{1/(\rho(F)+\epsilon)}\right\}(\rho(F)+1)(n+m+\lambda)^{\rho(F)/(\rho(F)+1)}.$$

It gives

$$\frac{(\rho(F)+1)^{\rho(F)+1}}{(\rho(F))^{\rho(F)}} T(F) \geq \limsup_{n+m+\lambda\to\infty} \left\{\frac{(\log^+ a_{nm\lambda} {r_1}^n {r_2}^m {r_3}^\lambda)^{\rho(F)+1}}{(n+m+\lambda)^{\rho(F)}}\right\}. \tag{3.12}$$

In order to prove reverse inequality, let us assume the right hand side of (3.10) is equal to β^* and consider the function

$$F^/(t,t^*,\zeta,\theta) = \sum_{n+m+\lambda=0}^{\infty} exp\left\{\ (\beta^*+\epsilon)^{1/(\rho(F)+1)}(n+m+\lambda)^{\rho(F)/(\rho(F)+1)}\right\}\ t^n {t^*}^m \zeta^\lambda \theta^{(n+m+\lambda)}$$

in place of $F^*(t,t^*,\zeta,\theta)$ in Theorem 3.1. Now using [4, Lemma 2] with $D = \frac{\rho(F)}{(\rho(F)+1)}$ and $C = (\beta^*+\epsilon)^{1/(\rho(F)+1)}$, for θ sufficiently close to 1, we obtain

$$log^+ M_{D_{\epsilon,R}}(\theta, F^/) < \frac{(\rho(F))^{\rho(F)}}{(\rho(F)+1)^{\rho(F)+1)}} \ (\beta^* + \epsilon) + O(1)\,(1-\theta)^{-\rho(F)}$$

Thus we have

$$M_{D_{\epsilon,R}}(\theta, F) < c_1 \theta^{\gamma(\epsilon)} + c_2 + exp\left\{\frac{(\rho(F))^{\rho(F)}}{(\rho(F)+1)^{\rho(F)+1)}} \ (\beta^* + \epsilon) + O(1)\,(1-\theta)^{-\rho(F)}\right\}$$

or

$$T(F) = \lim_{\theta \to 1} sup \left\{\frac{log^+ M_{D_{\epsilon,R}}(\theta, F)}{(1-\theta)^{\rho(F)}}\right\} \leq \frac{(\rho(F))^{\rho(F)}}{(\rho(F)+1)^{\rho(F)+1)}} \ (\beta^* + \epsilon)$$

or

$$\beta^* \geq \frac{(\rho(F)+1)^{\rho(F)+1}}{(\rho(F))^{\rho(F)}} T(F) \tag{3.13}$$

(3.12) and (3.13) together gives the required result.

If $T(F) = 0, then\ \ F(t, t^*, \zeta)$ is of order at most $\rho(F)$ and growth $(\rho(F), 0)$. Similarly, if $T(F) = \infty$, its growth is $(\rho(F), \infty)$. Hence the proof is completed.

Conclusion. Using the function theoretic method with Laplacian type integral operator (and inverse) between the analytic function $F(t, t^*, \zeta) \in C^3$ and harmonic function $u \in R^4$ $[3, p. 147]$ we can study the growth properties of solutions of Poisson's equation (1.3) with the help of Theorem A, Theorem 3.1 and Theorem 3.2.

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