Coefficients Characterization of Generalized Growth Parameters of Functions Analytic in Finite Disc

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Abstract. It has been noticed that the functions analytic on the finite disc having same order but infinite type, the concept of the type does not give the precise information about their growth. To overcome this problem the concept of type with respect to proximate order has been defined which is known as generalized type. Similarly, generalized lower type has been introduced. These parameters are known as generalized growth parameters. In this paper, we shall study the generalized growth parameters of a function analytic in finite disc D_R with respect to a proximate order $\rho_R(r)$, in terms of the coefficients a_n occurring in the Taylor series expansion of *f*.

Keywords and Phrases. Proximate Order, generalized growth, Taylor series.

1. Introduction. Several authors (see [1-7]) studied the growth parameters of functions analytic on the disk in terms of coefficients occurring in its Taylor series expansion and polynomial approximation errors. It is known that $\left({Rr}/{(R-r)}\right)^{\rho_R(r)}$ is a monotonically increasing function of r for $0 < r_0 < r < R$, therefore, we define a single valued function $\theta(x)$ for $x > x_0,$ such that

$$x = \left(\frac{Rr}{R-r}\right)^{\rho_R(r)+1} \Leftrightarrow \left(\frac{Rr}{R-r}\right) = \theta(x). \qquad (1.1)$$

To prove our main results, we need the following lemma.

Lemma 1.1. For the function $\theta(t)$ defined above the following relations hold

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$$\lim_{t \to \infty} \left(\frac{d[\log \theta(t)]}{d[\log t]} \right) = \frac{1}{\rho + 1}$$

and for $0 < \eta < \infty$

$$\lim_{t \to \infty} \left(\frac{\theta(\gamma t)}{\theta(t)} \right) = \gamma^{\frac{1}{\rho+1}}$$

Proof. Using the definition of proximate order and relation (1.1), we have

$$\frac{d[\log \theta(t)]}{d[\log t]} = \frac{1}{\rho + \rho_R^{/}(r)\,(R - r)\,log\left(\frac{R-r}{R}\right)}$$

$$\Leftrightarrow \frac{d[\log \theta(t)]}{d[\log t]} = \frac{1}{\rho + 1}$$

Now for a given $\epsilon > 0 \; and \; t > t_0$ we obtain

$$\int_t^{\gamma t} \left(\frac{1}{\rho + 1} - \epsilon \right) d[\log t] < \int_t^{\gamma t} d[\log \theta(t)] < \int_t^{\gamma t} \left(\frac{1}{\rho + 1} + \epsilon \right) d[\log t]$$

$$or \; \left(\frac{1}{\rho + 1} - \epsilon \right) \log \gamma < \log \frac{\theta(\gamma t)}{\theta(t)} < \left(\frac{1}{\rho + 1} + \epsilon \right) \log \gamma$$

$$or \quad \gamma^{\left(\frac{1}{\rho+1} - \epsilon\right)} < \frac{\theta(\gamma t)}{\theta(t)} < \gamma^{\left(\frac{1}{\rho+1} + \epsilon\right)}$$

$$or \quad \lim_{t \to \infty} \left(\frac{\theta(\gamma t)}{\theta(t)} \right) = \gamma^{\frac{1}{\rho+1}}.$$

2. Main Results

Now we will prove the following theorems:

Theorem 2.1. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$,analytic in$D_R$, having order ρ and proximate order$\rho_R(r)$. Then the type $T_R^*$of f with respect to the proximate order $\rho_R(r)$, is given by

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$$\frac{T_R^*}{M} = \limsup_{n \to \infty} \left[\frac{\theta(n) log^+ |c_n| R^n}{n}\right]^{\rho+1} \tag{2.1}$$

where

$$M = \frac{\rho^\rho}{(\rho+1)^{\rho+1}}.$$

Proof. For all r sufficiently close to R and for every $\epsilon > 0$, using the relation

$$T_R^* = \limsup_{r \to R} \frac{\log M(r,f)}{\left({Rr}/{(R-r)}\right)^{\rho_R(r)}}$$

We obtain

$$\log M(r,f) < (T_R^* + \epsilon)\left(\frac{Rr}{R-r}\right)^{\rho_R(r)}$$

Now by using Cauchy's inequality, we get

$$log^+|c_n| < (T_R^* + \epsilon)\left(\frac{Rr}{R-r}\right)^{\rho_R(r)} - nlogr \tag{2.2}$$

The right hand side of (2.2) is estimated at

$$\left(\frac{Rr}{R-r}\right)^{\rho_R(r)} = \frac{n}{\rho(T_R^* + \epsilon)}.$$

Then $v(x) \to R \; as \; n \to \infty$. Putting

$$P(n) = \frac{R}{R + \rho\, v(n)}$$

In view of (2.2), for all sufficiently large n, we get

$$log^+|c_n|R^n < \frac{(T_R^* + \epsilon)^{P(n)} n^{(1-P(n))}}{1 - P(n)} - n \log v(n) + n \log R$$

$$\Rightarrow \frac{\theta(n) log^+|c_n|R^n}{n} < \frac{(T_R^* + \epsilon)^{P(n)} \theta(n)}{\rho^{(1-P(n))} n^{P(n)}} \left(1 - \frac{\rho n^{P(n)} \log\left(\frac{v(n)}{R}\right)}{\rho (T_R^* + \epsilon)^{P(n)}}\right)$$

Since

$$\lim_{n\to\infty} \frac{\theta(n)}{n^{P(n)}} \to 1, \qquad \lim_{n\to\infty} \left(\frac{n^{P(n)} \log\left(\frac{\upsilon(n)}{R}\right)}{\rho(T_R^* + \epsilon)^{P(n)}} \right) \to -1 \tag{2.3}$$

and

$$\rho_R(r) \to R \; and \; r \to R,$$

(2.2) gives that

$$\frac{T_R^*}{M} \geq \limsup_{n\to\infty} \left[\frac{\theta(n) log^+|c_n|R^n}{n}\right]^{\rho+1} \tag{2.4}$$

In order to prove the reverse inequality, let αbe defined by the equation

$$\limsup_{n\to\infty} \left[\frac{\theta(n) log^+|c_n|R^n}{n}\right]^{\rho+1} = \frac{\alpha}{M}.$$

Thenfor all r sufficiently close to R and for every $\beta > \alpha$,

$$|c_n|R^n < \exp\left[\frac{n(1+\rho)\beta^{\frac{1}{1+\rho}}}{\rho^{\left(\frac{\rho}{1+\rho}\right)}} + n \log\frac{r}{R}\right].$$

In view of (1.1) and Lemma 1.1, we obtain

$$|c_n|R^n < \exp\left[\frac{n(1+\rho)}{\rho\, \theta\left(\frac{n}{\beta\, \rho}\right)} - n\left(\frac{R-r}{Rr}\right)\right].$$

Thus, the maximum term $\mu(r)$ of f for $|z| = r$ satisfies

$$\log \mu(r) < \max_{n\geq 0} \left[\frac{n(1+\rho)}{\rho\, \theta\left(\frac{n}{\beta\, \rho}\right)} - n\left(\frac{R-r}{Rr}\right)\right].$$

The right hand side is estimated at

$$n = \left[\beta\rho\left(\frac{Rr}{R-r}\right)^{\rho_R(r)+1}\right]$$

It gives

$$\frac{\log\mu(r)}{\left(\frac{Rr}{R-r}\right)^{\rho_R(r)}} < \beta$$

Now applying the limit as $r \to R$, we get

$$T_R^* \leq \beta$$

Since $\beta > \alpha$, we have

$$T_R^* \leq \alpha. \qquad (2.5)$$

On combining (2.4) and (2.5) the required result is immediate.

Theorem 2.2. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$, analytic in D_R, having order ρ $(0 < \rho < \infty)$ and proximate order $\rho_R(r)$ be such that $\psi(n) = \left|\frac{c_{n-1}}{c_n}\right|$ forms a non decreasing function of n for $n > n_0$. Then the lower type t_R^* of f, with respect to the proximate order $\rho_R(r)$ is given by

$$\frac{t_R^*}{M} = \liminf_{n \to \infty} \left[\frac{\theta(n) log^+|c_n|R^n}{n}\right]^{\rho+1}$$

Proof. Since $\psi(n)$ forms a non decreasing function of n, it can be seen that $\psi(n+1) > \psi(n)$ for infinitely many values of n and $\rho > 0, \psi(n) \to 1$ $as\ n \to \infty$, where $\psi(n+1) > \psi(n)$, the maximum term $\mu(r)$ and the central index $\nu(r)$ of f, for $\psi(n) \leq r \leq \psi(n+1)$, are given by

$$\mu(r) = |c_n| r^n and\ \nu(r) = n.$$

Let $0 < t_R^* < \infty$. Using the relation

$$t_R^* = \liminf_{r \to R} \frac{\log M(r,f)}{\left(Rr/(R-r)\right)^{\rho_R(r)}}$$

For given $\epsilon > 0$ and for all *r* sufficiently close to *R*, we get

$$\log \mu(r) > (t_R^* - \epsilon)\left(\frac{Rr}{R-r}\right)^{\rho_R(r)}$$

If $c_{n_1} z^{n_1}$ *and* $c_{n_2} z^{n_2}$ are consecutive maximum terms of *f*, and $n_1 \leq n \leq n_2 - 1$, then $\psi(n_1+1) = \psi(n_1+2) = \cdots \ldots \ldots \ldots \ldots \ldots = \psi(n_2)$ *and* $|c_n| r^n = |c_{n_1}| r^{n_1}$ *for* $\frac{r}{R} = \psi(n+1)$.

Therefore,

$$log^+|c_n| + n \log R\, \psi(n+1) > (t_R^* - \epsilon)\left(\frac{R - \psi(n+1)}{R\psi(n+1)}\right)^{-\rho\psi(n+1)}$$

Since $-\log Rx \geq \frac{1}{x} - \frac{1}{R}$ *for* $x > 0$, we have

$$\frac{\theta(n) log^+|c_n| R^n}{n} > \frac{(t_R^* - \epsilon)\theta(n)}{n}\left(\frac{R - \psi(n+1)}{R\psi(n+1)}\right)^{-\rho\psi(n+1)} - \frac{n}{(t_R^* - \epsilon)}\left(\frac{R - \psi(n+1)}{R\psi(n+1)}\right) \quad (2.6)$$

Suppose that

$$n(x) = \left(\frac{R-x}{Rx}\right)^{-\rho(x)} + \frac{n}{(t_R^* - \epsilon)}\left(\frac{R-x}{Rx}\right)$$

The maximum value of the function *n(x)* occurs at a point $x_1 = x_1(n)$ given by, for n sufficiently large,

$$\left(\frac{R-x}{Rx}\right)^{-\rho(x)-1} = \frac{n}{(t_R^* - \epsilon)(\rho + o(1))}$$

Using the definition of $\theta(x)$, we obtain

$$\left(\frac{R-x}{Rx}\right)^{-1} = \theta\left[\frac{n}{(t_R^* - \epsilon)(\rho + o(1))}\right].$$

For sufficiently large value of n, we get

$$\inf_{0<x<R} n(x) = \left[\frac{n}{(t_R^* - \epsilon)}\big(\rho + o(1)\big)\theta\left[\frac{n}{(t_R^* - \epsilon)(\rho + o(1))}\right] + \frac{n}{(t_R^* - \epsilon)\theta\left[\frac{n}{(t_R^*-\epsilon)(\rho+o(1))}\right]}\right]$$

$$= \frac{n}{(t_R^* - \epsilon)}\theta\left[\frac{n}{(t_R^* - \epsilon)(\rho + o(1))}\frac{\big(1+\rho+o(1)\big)}{\big(\rho+o(1)\big)}\right].$$

In view of (2.6) and Lemma 1.1, we get

$$\liminf_{n\to\infty}\left[\frac{\theta(n)log^+|c_n|R^n}{n}\right]^{\rho+1} \geq \frac{(\rho+1)^{\rho+1}}{\rho^\rho} t_R^* \tag{2.7}$$

Inequality (2.7) obviously holds if $t_R^* = 0$.

To prove the equality in (2.7) we shall prove that strict inequality cannot hold in (2.7). For, if it holds, then there exists a number $\delta > t_R^*$, such that

$$\liminf_{n\to\infty}\left[\frac{\theta(n)log^+|c_n|R^n}{n}\right]^{\rho+1} = \frac{(\rho+1)^{\rho+1}}{\rho^\rho}\delta.$$

Let δ_1 be such that $\delta > \delta_1 > t_R^*$. Then, for all n sufficiently large

$$log^+|c_n|R^n > \frac{n}{\theta(n)}\frac{(1+\rho)}{\rho^{\frac{\rho}{(1+\rho)}}}\delta_1^{\frac{1}{(1+\rho)}}.$$

For all r, sufficiently close to R and sufficiently large

$$log^+M(r,f) > \frac{n}{\theta(n)}\frac{(1+\rho)}{\rho^{\frac{\rho}{(1+\rho)}}}\delta_1^{\frac{1}{(1+\rho)}} + n\log\frac{r}{R}$$

$$= \frac{n}{\theta(n)}\frac{(1+\rho)}{\rho^{\frac{\rho}{(1+\rho)}}}\delta_1^{\frac{1}{(1+\rho)}} - n\log\left(\frac{R-r}{R}\right).$$

Assume that $n = \left[\delta_1\rho\left(\frac{R-r}{Rr}\right)^{-\rho_R(\mathrm{r})-1}\right]$, then, in view of definition of $\theta(n)$, we get

$$\log M(r,f) > \frac{n}{\rho\theta\left(\frac{n}{\delta_1\rho}\right)} = \delta_1\left(\frac{Rr}{R-r}\right)^{\rho_R(\mathrm{r})}$$

so that $t_R^* \geq \delta_1$, which is a contradiction. Hence the proof is completed.

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